GENERAL SOLUTION TO UNIDIMENSIONAL HAMILTON-JACOBI EQUATION

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Abstract

A method for finding the general solution to the partial differential equations: $F(u_x, u_y) = 0$; $F(f(x)\, u_x, u_y) = 0$ (or $F(u_x, h(y)\, u_y) = 0$) is presented, founded on a Legendre like transformation and a theorem for Pfaffian differential forms. As the solution obtained depends on an arbitrary function, then it is a general solution. As an extension of the method it is obtained a general solution to PDE: $F(f(x)\, u_x, u_y) = G(x)$, and then applied to unidimensional Hamilton-Jacobi equation.

Key words: Partial Differential Equations of First Order; Nonlinear PDEs; Hamilton-Jacobi Equation.

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1 Introduction

The practical and conceptual importance of the Hamilton-Jacobi equation can be extensively pointed: as a fundamental concept in classical mechanics [1]; as a practical tool for solving differential equation [2]; as a base to quantization [4]; as an approximation of zero order in the WKB method [5]; ...

The solutions of the Hamilton-Jacobi equation are usually determined as integral solutions through the method of separation of variables. But general solutions of this equation are more important either by its conceptual meaning [6], as by the characteristic of a general solution (an infinity of integral solutions).

Unfortunately, due to the nonlinearity of this equation till now there is no available technique to determine a general solution [7, 8] in most problems.

The purpose of this article is solve this centenary problem. The method solves the unidimensional problem changing an insurmountable problem by an eventual practical problem of solving an algebraic equation. Although to this algebraic problem the actual advanced computational numerical techniques can be applied.

The procedure applied to solve this problem ia an extension of that developed for PDEs, linear or not, of one of the types:

$$F(p,q)=0; \quad F(f(x)\,p,q)=0 \quad (or \quad F(p,h(y)\,q)=0),$$

where $p=\partial u/\partial x$, $q=\partial u/\partial y$ and $u=u(x,y)$ [10, 11].

First this method of obtaining a general solution will be summarized, then its extension to the Hamilton-Jacobi equation will be developed.

2 General Solution to the PDE $\mathbf{F(p,q)=0}$

There are few books that accost the subject of partial differential equations of first order and also the methods presented as the Charpit's, or the characteristic's, or separation of variables one, or another techniques only supplies integral (complete) solutions to this type of equation unless it is a linear one (Lagrange PDE) [7, 8, 3, 9]. In the method developed an infinity of integral solutions are obtained. Because it furnishes a general solution in an implicit form and in some cases a explicit one. In the non linear PDEs the general solution is really obtained exchanging an impossible problem by the solution of algebraic equations.

The procedure developed is based on a Legendre like transformation and the use of a theorem that gives the condition of integrability for Pfaffian differential forms [7].

Consider a PDE of first order

$$F(p,q)=0, \tag{1}$$

where $p=\partial u/\partial x$, $q=\partial u/\partial y$ and $u=u(x,y)$. The Pfaffian differential form for u is

$$du=p\,dx+q\,dy.$$

Applying a Legendre like transformation results that

$$d(xp + yq) - du - xdp - ydq = 0.$$

As $dF = F_p dp + F_q dq = 0$, then

$$dp = -(F_q/F_q)dq,$$

therefore

$$d(xp + yq) - du + \left(x\frac{F_q}{F_p} - y \right) dq = 0 \; . \tag{2}$$

Since this is a Pfaffian differential form then the following theorem can be applied [7]:

Theorem

A necessary and sufficient condition that the Pfaffian differential equation $\vec{X} \cdot \vec{r} = 0$ should be integrable is that $\vec{X} \cdot rot\, \vec{X} = 0$.

Therefore from the theorem the condition of integrability applied to equation (2) results in

$$\vec{X} \cdot rot\, \vec{X} = -\left(\frac{\partial}{\partial(xp + yq)} + \frac{\partial}{\partial u} \right) \left(x\frac{F_q}{F_p} - y \right) = 0,$$

which integrated gives

$$u - xp - yq = \phi(q). \tag{3}$$

The use of this result in equation (2) supplies the additional condition

$$\left(x\frac{F_q}{F_p} - y \right) = -\phi'(q). \tag{4}$$

Then the general solution of de PDE is given by the equation (3) where q is determined by equation (4) for every choice of the arbitrary function $\phi(q)$. This is a general solution since it has an arbitrary function $\phi(q)$, i.e., for each form of $\phi(q)$ the equations (4) and (1) results in a system of algebraic equations that determines $q = q(x, y)$ and $p = p(x, y)$, which gives a particular solution to the PDE when substituted in equation (3).

In some problems it can be writen explicitly $p = f(q)$ (or $q = g(p)$) and the general solution of the PDE from equation (3) now stay as

$$u = x\, f(q) + yq + \phi(q), \tag{5}$$

and the integration condition - equation (4) - turns in

$$xf_q + y = -\phi'(q), \tag{6}$$

which determines the variable $q = q(x, y)$ for every choice of the arbitrary function $\phi(q)$.

Let consider, as a first example, the equation

$$F(p,q) = p - Bq + A = 0,$$

where p can turn out explicit and A, B are constants. Therefore $p = f(q) = Bq - A$, and $f_q = B$ then the solution given by equation (5) is

$$u = xp + yq - \phi(q) = x(Bq - A) + yq - \phi(q),$$

where it must be obtained $q = q(x, y)$ from equation (6).

The equation (6) gives

$$xf_q + y = Bx + y = \phi'(q),$$

then

$$q = \psi(Bx + y)$$

and the general solution is

$$u = Ax + (Bx + y)\,\psi(Bx + y) - \phi^{-1}(\psi(Bx + y)) = Ax + \Phi(Bx + y).$$

This is the same result as that given by the method for a Legendre PDE [7].

As another example consider the non linear PDE $p^n - q^m = A$, where A is constant. Then $p = (A + q^m)^{1/n} = f(q)$, and

$$f_q = \frac{m}{n}(A + q^m)^{(1-n)/n} q^{m-1}$$

therefore from equation (5) the general solution is

$$u = x(A + q^m)^{1/n} + yq - \phi(q),$$

with $q = q(x, y)$ obtained from equation (6) rewritten as

$$\phi'(q) = \frac{mx}{n}(A + q^m)^{(1-n)/n} q^{m-1} + y.$$

Therefore each arbitrary choice of $\phi(q)$ provides an integral solution.

3 General Solution to the PDE $\mathbf{F(f(x)\,p, q) = 0}$ (or $\mathbf{F(p, h(y)\,q) = 0}$)

Following an identical procedure as that of the last section, from the PDE $F(f(x)p, q) = 0$ it can be written as $p = G(q)/f(x)$. The substitution of this result in the differential form associated and the application of a Legendre like transformation results in

$$du = d[H(x)\,G(q) + yq] - [H(x)\,G'(q) + y]dq,$$

where $H(x) = x/f(x)$.

As this is a Pfaffian differential form then the same theorem of the last section can be applied, resulting in the condition of integrability

$$H(x)G\,'(q) + y = \phi'(q). \tag{7}$$

Therefore the general solution is

$$u = H(x)G(q) + yq - \phi(q), \tag{8}$$

where $\phi(q)$ is an arbitrary function, which once selected provides the value of the variable $q = q(x, y)$ by the equation (7).

In similar manner it can be obtained the general solution of the PDE

$$F(p, h(y)\, q) = 0.$$

If

$$q = G(p)/h(y)$$

then the general solution is given by

$$u = xp + G(p)H(y) - \phi(p),$$

where $H(y) = y/h(y)$, with the integrability condition

$$\phi'(p) = G\,'(p)H(y) + x.$$

4 General Solution To Unidimensional Hamilton-Jacobi Equation

Considere the most general Hamilton-Jacobi equation for a conservative unidimensional non relativistic mechanical system

$$a(x)p^2 + V(x) - q = 0, \tag{9}$$

where $p = \partial S/\partial x$ and $q = \partial S/\partial t$.

As $S = S(x, t)$ then the differential form can be written as

$$dS = pdx + qdt = d(px + qt) - xdp - tdq, \tag{10}$$

where a Legendre like transformation was applied.

The use of p from (9) in the above equation results in

$$dS = d\left(\frac{x\sqrt{a(q-V)}}{a} + qt\right) - \frac{x(a'V - aV' - qa')}{2\sqrt{a(q-V)}}dx - \left(t + \frac{x}{2\sqrt{a(q-V)}}\right)dq, \tag{11}$$

where $a' = da/dx$ and $V' = dv/dx$, yielding

$$S(x, t) = x\sqrt{(q-V)/a} + qt - F(x, q), \tag{12}$$

sendo F tal que

$$\frac{\partial F}{\partial q} = t + \frac{x}{2\sqrt{a(q-V)}}\,, \tag{13}$$

$$\frac{\partial F}{\partial x} = \frac{x(a'V - aV' - qa')}{2\sqrt{a(q-V)}} \equiv H(x,q)\,. \tag{14}$$

The integration of the equation (14) furnishes $F = \int H(x,q)dx + G(q)$, where G is an arbitrary function. This result applied in equation (13) gives an equation that defines the variable $q = q(x,t)$, for every arbitrary choice of the function G:

$$\int \frac{\partial H}{\partial q} dx + G'(q) = y + \frac{x}{2\sqrt{a(q-V)}}\,. \tag{15}$$

As this solution contains an arbitrary function then $S = S(x,t)$ given by (12) is a general solution.

Its interesting emphasize that to obtain the solution by the method of separation of variables to this Hamilton-Jacobi equation it is imposed that $q = constante$ (i.e., $dq = 0$, $S(x,t) = W(x) + C(t)$).

5 Examples

As the first example consider the Hamilton-Jacobi equation that describes a free particle $ap^2 - q = 0$ ($a = constante$). The solution from (12) is

$$S = x\sqrt{q/a} + qt - F.$$

where the function F is obtained from the solution of the system composed by (13) e (14)

$$F\,'(q) = t + \frac{x}{2\sqrt{aq}}\,.$$

The last equation furnishes the variable $q = q(x,t)$ for every choice of the arbitrary function F. For example, if $F = Cq$ then $q = x^2/4a(C-t)^2$, therefore $S(x,t) = x^2/4a(C-t)$. This is the same solution obtained using the movement data of the particle [12], which is unnecessary in our method.

The solution $S(x,t) = x\sqrt{C/a} + Ct$ given by method of separation of variables applied to this Hamilton-Jacobi equation is obtained making $dq = 0$ in (11).

Let consider as another example the Hamilton-Jacobi equation of a simple harmonic oscillator

$$p^2 + x^2 - q = 0.$$

The solution from (12) is

$$S(x,t) = qt + \frac{x}{2}\sqrt{q - x^2} + \frac{q}{2}\sin^{-1}\frac{x}{\sqrt{q}} - G(q),$$

where $q = q(x,t)$ is fixed by each choice of the arbitrary function G from

$$t + \frac{1}{2} x \sqrt{q - x^2} = G'(q) + \frac{q}{2}\, sen^{-1}\left(\frac{x}{\sqrt{q}}\right).$$

The solution form the method of separation of variables $(q = C)$ is

$$S(x,t) = \frac{1}{2} x \sqrt{C - x^2} + Ct - \frac{C}{2}\, sen^{-1}\left(\frac{x}{\sqrt{C}}\right).$$

6 FINAL REMARKS

The procedure developed to solve the one dimensional Hamilton-Jacobi equation is an extension of that presented in section 2 and 3 [10, 11]. The integrability condition for Pfaffian differential forms (3) [7] imply in the equations (13) and (14).

The extension of this method to Hamilton-Jacobi equations two and tridimensional and a general formulation for this type of PDEs can be a later approach.

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